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Dynamics on random ultrametric spaces

William G Motyl

Department of Physics, Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544, USA

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Abstract. The problem of a random walk on an ultrametric space which is characterised by random branching and non-degenerate sites is considered. Results are compared with more restrictive solutions.

1. Introduction

A number of recent papers have discussed stochastic dynamics on ultrametric spaces [1-7]. In these systems a particle hops from site to site by thermal activation over a hierarchy of barriers. Depending on the scaling of the barriers, the autocorrelation function displays a range of non-exponential behaviour including a temperature-dependent power-law decay. The qualitative nature of these systems (large number of free energy minima, many different timescales) and the nature of these results (slow relaxation) suggest the usefulness of ultrametric models for understanding disordered systems. This is discussed, for example, by Blumen *et al* [4], who consider the use of ultrametric diffusion to model transport in random media. Earlier papers [1, 2] restricted their consideration to degenerate sites in highly regular spaces, but a degree of randomness in the branching as well as in the free energies of the sites would clearly be appropriate. Later papers [3-6] deal with the more general case in which branching may differ from level to level. Kumar and Shenoy [5] consider the introduction of randomness in the branchings within the levels and suggest that such randomness has little effect on the form of the autocorrelation function for diffusion through such systems. Bachas and Huberman [7] obtain a solution for arbitrary branching and are able to average it in closed form over independent distributions for the branching in the limit of many-levelled spaces with linearly increasing barriers. They indicate that the exponent in the power-law decay of the autocorrelation function that arises in this case is unchanged by the introduction of the randomness. The present paper extends these results by solving quite generally the case of ultrametric spaces with independent random branching and by giving detailed consideration to the effect of non-degenerate sites. In the case of random branching the solution permits direct numerical evaluation of the exact average autocorrelation for finite-size spaces. In § 2 the general solution for arbitrarily branched spaces is presented. Averaging over distributions of the branching is performed in § 3. Up to this point the sites are taken to have equal thermodynamic weight. This restriction is removed in § 4. Finally, results are compared to those that pertain to uniform branching and degenerate sites. With respect to randomness in the branching and to non-degeneracy of the sites, the qualitative features of the autocorrelation are found to be robust.

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2. Solution for arbitrary branching

We consider a space of sites that has the following structure. At the first level of branching, the sites are partitioned into clusters. An ultrametric distance is defined so that any two sites belonging to different clusters are separated by a distance of n. These clusters will be said to be of order $(n-1)$ and are then in turn partitioned into subclusters of order $(n-2)$ and so on down to clusters of order 0, i.e. single sites.

The dynamics are described by a stochastic process

$$\mathrm{d}\boldsymbol{x}/\mathrm{d}t = \boldsymbol{W}\cdot\boldsymbol{x}(t) \tag{1}$$

where $x_i(t)$ is the probability of finding the system at site i at time t. For any specific tree, the transition matrix $\boldsymbol{W}$ is given by

$$W_{ij} = \begin{cases} \varepsilon_k & g(i,j) = k,\ k \neq 0 \\ d_i & i = j. \end{cases} \tag{2}$$

Here $g(i,j)$ is the ultrametric distance between sites i and j. The off-diagonal entries ε_k are the hopping rates for hops of distance k. The diagonal entries d_i are determined by the condition that probability be conserved,

$$\sum_i W_{ij} = 0. \tag{3}$$

Hence

$$d_i = -\sum_{r=1}^{n} n_{ir}\varepsilon_r. \tag{4}$$

Here n_{ir} is the number of sites at distance r from site i. The autocorrelation function is then $\boldsymbol{x}(0)\cdot\boldsymbol{x}(t)$, and may be found by expanding $\boldsymbol{x}$ in eigenvectors of $\boldsymbol{W}$. These eigenvectors are readily found by making the appropriate ansatzë. Let $\mathscr{A}_\mu$ be a cluster of m_μ sites. Further, let $\mathscr{A}_\mu$ be partitioned into α_μ subclusters, $\mathscr{A}_{\mu\nu}$. An appropriate trial vector corresponding to $\mathscr{A}_\mu$ is

$$x_i^{(\mathscr{A}_\mu)} = \begin{cases} 0 & i \notin \mathscr{A}_\mu \\ b_\nu & i \in \mathscr{A}_{\mu\nu}. \end{cases} \tag{5}$$

Corresponding to each $\mathscr{A}_\mu$ of order $\geqslant 1$ we find an $(\alpha_\mu - 1)$-dimensional eigenspace. With the inclusion of a one-dimensional eigenspace corresponding to the zero eigenvalue, the decomposition of $\boldsymbol{W}$ is complete.

Rather than describe the individual eigenvectors it is convenient to introduce projection operators:

$$P_{ij}^{\prime(\mathscr{A}_\mu)} = \begin{cases} 0 & i \notin \mathscr{A}_\mu \text{ or } j \notin \mathscr{A}_\mu \\ 1 - 1/m_\mu & i \in \mathscr{A}_\mu,\ j = i \\ -1/m_\mu & i \in \mathscr{A}_\mu,\ j \in \mathscr{A}_\mu,\ i \neq j \end{cases} \tag{6}$$

and

$$P_{ij}^{(\mathscr{A}_\mu)} = P_{ij}^{\prime(\mathscr{A}_\mu)} - \sum_{\nu=1}^{\alpha_\mu} P_{ij}^{\prime(\mathscr{A}_{\mu\nu})}. \tag{7}$$

The $P^{(\mathscr{A}_\mu)}$ project out the eigenvectors corresponding to respective $\mathscr{A}_\mu$. With these in hand it is a simple matter to determine the eigenvalues:

$$\lambda^{(\mathscr{A}_\mu)} = -\left(\sum_{r=0}^{j} n_{ir}\varepsilon_j + \sum_{r=j+1}^{n} n_{ir}\varepsilon_r \right) \tag{8}$$

for $i \in \mathscr{A}_\mu$ and $\mathscr{A}_\mu$ of order j. (The second sum is taken to be 0 when $j = n$.) Finally,

$$\boldsymbol{x}(0) \cdot \boldsymbol{x}(t) = \boldsymbol{x}(0) \cdot \sum_{\mathscr{A}_\mu} \exp(-\lambda^{(\mathscr{A}_\mu)} t) \boldsymbol{P}^{(\mathscr{A}_\mu)} \cdot \boldsymbol{x}(0) \tag{9}$$

up to a constant term equal to one over the number of sites. In the case where $\boldsymbol{x}(0)$ describes the definite occupation of a single given site, i.e. $x_j(0) = \delta_{ij}$, the result may be simplified. Let site i be characterised by a sequence of numbers $w_0, w_1, \ldots, w_n$ where each w_r is the number of sites in the cluster of order r of which the initial site, i, is a member. Then, retaining a factor of $w_0 = 1$ for reasons which will become clear in § 4,

$$\boldsymbol{x}(0) \cdot \boldsymbol{x}(t) = \sum_{i=1}^{n} w_0 \left(\frac{1}{w_{i-1}} - \frac{1}{w_i} \right) \exp\left[-\left(w_i\varepsilon_i + \sum_{j=i+1}^{n} (w_j - w_{j-1})\varepsilon_j \right) t \right] + \frac{1}{w_n} \tag{10}$$

in agreement with Bachas and Huberman [7].

3. Average over randomly branched spaces

Consider an ensemble of ultrametric spaces of order n with each α_u chosen independently according to some given distribution. Averaging is to be performed over initial sites, $x_j = \delta_{ij}$, chosen at random from this ensemble. The quantity of interest is

$$A = \frac{\sum_{w=1}^{\infty} P_n(w) A_n(w)}{\sum_{w=1}^{\infty} w P_n(w)} \tag{11}$$

where $A_n(w)$ is the autocorrelation summed over all initial sites in all spaces of n levels with w sites and $P_n(w)$ is the probability that a space of n levels has precisely w sites. Once again we drop the constant term. $P_n(w)$ and $A_n(w)$ will be evaluated recursively in n. The calculation of $P_n(w)$ in fact constitutes a famous problem that first arose in 1874, namely to determine the likelihood of a given family dying out in successive generations. The solution of this problem, as well as a brief history, is given by Harris [8]. Apparently, this solution has been rediscovered numerous times over the years (perhaps most recently by the author) and is recounted here as a clear illustration of the method used to solve for $A_n(w)$. We proceed to count the $(n+1)$-level spaces by counting the number of ways in which each may be constructed by appending n-level spaces to one-level spaces (figure 1):

$$P_{n+1}(w) = P_1(1) P_n(w) + \sum_{w_1} \sum_{w_2} P_1(2) P_n(w_1) P_n(w_2) \delta_{w, w_1 + w_2} + \ldots$$

$$+ \sum_{w_1} \ldots \sum_{w_i} P_1(i) \left(\prod_{j=1}^{i} P_n(w_j) \right) \delta_{w, \sum_{j=1}^{i} w_j} + \ldots . \tag{12}$$

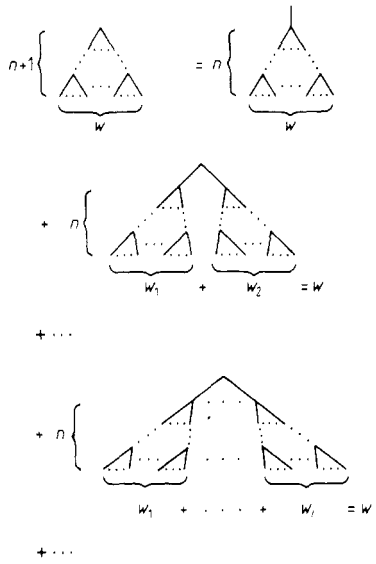

Figure 1. Diagram illustrating the method of counting branches.

This expression is more compactly expressed with the introduction of generating functions:

$$\tilde{P}_n(s) = \sum_{w=1}^{\infty} P_n(w) s^w. \tag{13}$$

Multiplying both sides of (12) by s^w and summing over w yields

$$\tilde{P}_{n+1}(s) = \sum_{i=1}^{\infty} P_1(i) \tilde{P}_n^i(s). \tag{14}$$

In addition, relation (14) implies recursive relations among the moments of the $P_n(w)$:

$$\langle w \rangle_{n+1} = \langle w \rangle_1 \langle w \rangle_n \tag{15}$$

and

$$\langle w^2 \rangle_{n+1} = \langle w \rangle_1 \langle w^2 \rangle_n + (\langle w^2 \rangle_1 - \langle w \rangle_1) \langle w \rangle_n^2. \tag{16}$$

Equations (15) and (16) iterate toward a fixed point in which the width of the distribution grows in proportion to the mean,

$$\sigma_n^2 = \frac{\sigma_1^2}{\langle w \rangle_1^2 - \langle w \rangle_1} \langle w \rangle_n^2. \tag{17}$$

A recursion relation for A_n may be derived in a similar manner. Let us build each $(n+1)$-level space of w sites out of n-level spaces having $w_1, w_2, \ldots$ sites. We sum the contributions to A_{n+1} as before noting that each n-level space contributes $A_n \exp[-(w_n - w_{n+1})\varepsilon_{n+1} t]$ to A_{n+1} and remembering to include the contribution from

the cluster of order $(n+1)$.

$$P_{n+1}(w)A_{n+1}(w)\exp(-w\varepsilon_{n+1}t)$$

$$= P_1(1)P_n(w)A_n(w)\exp(-w\varepsilon_{n+1}t) + P_1(2)\sum_{w_1,w_2=1}^{\infty} P_n(w_1)P_n(w_2)\delta_{w,w_1+w_2}$$

$$\times\left[A_n(w_1)w_1\exp(-w_1\varepsilon_{n+1}t) + A_n(w_2)w_2\exp(-w_2\varepsilon_{n+1}t) + \sum_{\text{sites}}\left(\frac{1}{w_i}-\frac{1}{w}\right)\right]$$

$$+\ldots+P_1(i)\sum_{w_1\ldots w_i}^{\infty} P_n(w_1)\ldots P_n(w_i)\delta_{w,w_1+\ldots+w_i}$$

$$\times\left[A_n(w_1)w_1\exp(-w_1\varepsilon_{n+1}t)+\ldots+A_n(w_i)w_i\exp(-w_i\varepsilon_{n+1}t)\right.$$

$$\left.+\sum_{\text{sites}}\left(\frac{1}{w_i}-\frac{1}{w}\right)\right]+\ldots. \tag{18}$$

Once again considerable simplification results from the introduction of generating functions. Let

$$\tilde{A}_n(s)=\sum_{w=1}^{\infty} P_n(w)A_n(w)s^w. \tag{19}$$

Then (18) becomes

$$A_{n+1}(s)=\left(\sum_{i=1}^{\infty} P_1(i)i\tilde{P}_n^{(i-1)}[s\exp(-\varepsilon_{n+1}t)]\right)\tilde{A}_n(s)$$

$$+\left(\sum_{i=1}^{\infty} P_1(i)(i-1)\tilde{P}_n^{i}[s\exp(-\varepsilon_{n+1}t)]\right). \tag{20}$$

Finally, to start the recursion,

$$A_1(s)=\sum_{i=1}^{\infty} P(i)(i-1)\exp(-\varepsilon_1 t)s^i. \tag{21}$$

It is apparent that (11), (20) and (21) constitute a solution that is amenable to direct numerical evaluation. Moreover, this formalism can treat spaces with different distributions chosen for each level although such spaces will not be considered here.

4. Non-degenerate sites

In considering states with different thermodynamic weights we observe with Paladin *et al* [2] that W_{ij} must be replaced by $W_{ij}g_j$ (for $i\neq j$) where g_i is the weight of site i and with Bachas and Huberman [7] that this weighting of sites may be achieved in effect by adding one more level to the ultrametric space with site i now represented as a cluster of g_i microstates. The present paper extends these considerations: we proceed by taking the limit in which the hopping rate among microstates of a given site approaches infinity so that the occupation of a given site, i, will always be simply g_i times the occupation of any of its microstates. Immediately after $t=0$ equation (10) will be valid with w_0 at each site no longer equal to one but equal to g_i at that site. We let $P'(g)$ be the distribution of thermodynamic weights of the sites. In order to preserve the average site to site transition rates we demand that $\langle g\rangle=1$ and permit

analytic continuation of the g_i from integer to real values. The autocorrelation is derived as before with two complications. We must now start the recursion for $\tilde{P}_n(s)$ with

$$\tilde{P}_1(s) = \sum_{i=1}^{\infty} P_1(i)\tilde{P}^{ri}(s) \tag{22}$$

and

$$\sum_{\text{sites}} \left(\frac{1}{w_i} - \frac{1}{w}\right) \to \sum_{\text{microstates}} g_j \left(\frac{1}{w_i} - \frac{1}{w}\right). \tag{23}$$

5. Discussion

The solution for the case of uniform branching may be recovered either from equation (10) by letting $w_i = \alpha^i$ or from equation (20) by letting $\tilde{P}_n(s) = s^{\alpha^n}$. When $\alpha = 2$, for example, the solution of Ogielski and Stein [1] is recovered. In comparing solutions for the autocorrelation we see that the effect of introducing random branching is to replace single exponentials by distributions of exponentials:

$$\exp(-\langle w\rangle_n \varepsilon_n t) \to \sum_{w=1}^{\infty} P_n(w) \exp(-w\varepsilon_n t). \tag{24}$$

It may be argued that this difference is qualitatively unimportant. Consider the terms in the autocorrelation for spaces branching uniformly at a rate of α per level:

$$x_n = \left(\frac{1}{\alpha^{n-1}} - \frac{1}{\alpha^n}\right) \exp\{-[\alpha^n(\varepsilon_n - \varepsilon_{n+1}) + \alpha^{n+1}(\varepsilon_{n+1} - \varepsilon_{n+2}) + \ldots]t\}. \tag{25}$$

Initially, x_n is smaller than x_{n-1} but it decays more slowly. Let t_n be the time at which x_n becomes equal to x_{n-1}. Then

$$t_n = \frac{\ln \alpha}{\alpha^{n-1}(\varepsilon_{n-1} - \varepsilon_n)} \tag{26}$$

corresponds to the timescale at which x_n becomes important. (Note that analogous consideration may be given to the timescale at which x_n *ceases* to be important.) Let us compare, at time t_n, x_n to the corresponding term in the randomly branching case with $\langle w\rangle = \alpha$. Each term in x_n will be multiplied by a factor which is an average of exponentials with the mean exponent equal to zero. For a given term, say the ith in x_n, the width of the distribution of exponents will be

$$\frac{(\varepsilon_{n+i-1} - \varepsilon_{n+i})\sigma_{n+i-1}}{(\varepsilon_{n-1} - \varepsilon_n)\langle w\rangle^{n-1}} \ln\langle w\rangle. \tag{27}$$

Considering first the case of linearly increasing barriers, i.e. $\varepsilon_n = \exp(-an)$, and noting equation (17), we see that the introduction of random branching multiplies the terms in the autocorrelation each by factors that are essentially identical at the times the given terms are most important. It is then to be expected that the only change in the autocorrelation will be a shift by a more or less constant factor. This result is borne out numerically (figure 2(*a*)). When the barriers increase faster than linearly, the distributions of exponents become narrower from term to term and the autocorrelation is expected to tend toward that of the uniformly branching result (figure 2(*b*)). It

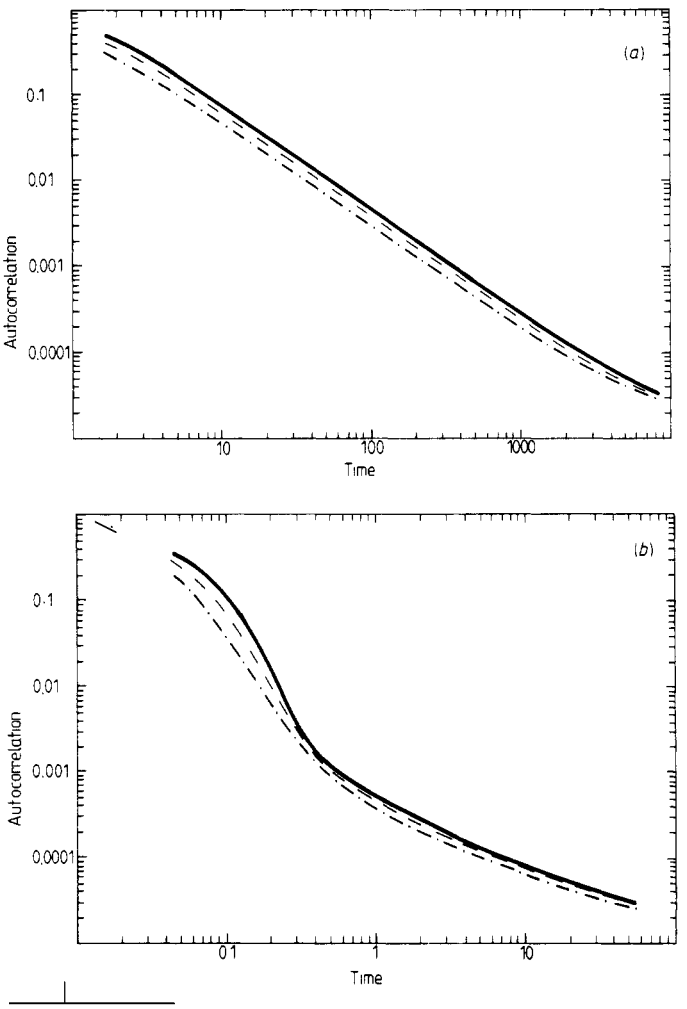

Figure 2. Plots of autocorrelation against time for ten-level trees with various branching distributions with a mean of three. The full curve is for uniform branching, the broken curve for a Poisson distribution and the chain curve for a geometric distribution. The autocorrelation for a flat distribution with 2, 3 or 4 branches equally likely coincides with that of the uniform branching case to within the resolution of the graphs. (*a*) Linearly increasing barriers, (*b*) quadratically increasing barriers (*c*) barriers that go as $n^{0.5}$.

would then seem that barriers increasing slower than linearly would lead to a solution that diverges from that of uniform branching. However, the solutions will diverge only to the point of approaching a constant ratio at large n (figure 2(*c*)). Note that the distribution that yields the greatest difference from uniform branching is the one with the longest tail, the geometric distribution.

In the case of linearly increasing barriers an additional argument may be made for the insignificance of randomness in the branching. Many results may be derived from simple scaling arguments. A typical example is the result of Ogielski and Stein [1] for the limiting form of the average ultrametric distance travelled,

$$\lim_{t\to\infty} \langle d(t)\rangle \sim (T/\Delta)\log t \tag{28}$$

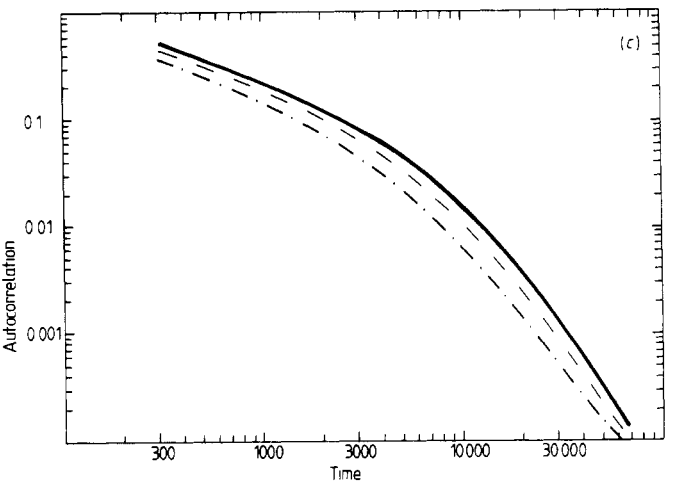

Figure 2. (continued)

in terms of the increase in barrier height, Δ. The distributions scale in width appropriately (in proportion to the mean) for the same scaling argument used to understand this result to remain valid.

Turning now to the case of non-degenerate sites, we note first that any initial broadening of the distributions due to equation (22) will become insignificant because the width will iterate toward the same fixed point, equation (17). Moreover, application of the central limit theorem after sufficiently many levels of branching yields

$$\sum_{\text{microstates}} g_i\left(\frac{1}{w_{n-1}}-\frac{1}{w_n}\right)\rightarrow\langle g^2\rangle\sum_{\text{sites}}\left(\frac{1}{w_{n-1}}-\frac{1}{w_n}\right) \tag{29}$$

which merely shifts the autocorrelation by a constant factor.

We have solved exactly the problem of diffusion in a randomly branching ultrametric space and have presented analytical arguments as well as numerical results indicating the qualitative robustness of the problem. Further arguments have been presented to extend these conclusions to spaces of non-degenerate sites.

Acknowledgment

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